

Q1

1

Given the identity

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

prove the following identities:

- (i) $\cos 2\theta \equiv \cos^2 \theta - \sin^2 \theta$
- (ii) $\cos 2\theta \equiv 1 - 2\sin^2 \theta$
- (iii) $\cos 2\theta \equiv 2\cos^2 \theta - 1$

$$\sin^2 \theta + \cos^2 \theta \equiv 1$$

$$\Rightarrow \cos^2 \theta \equiv 1 - \sin^2 \theta$$

$$\Rightarrow \sin^2 \theta \equiv 1 - \cos^2 \theta$$

[4]

save my exams

(i) Let $A = B = \theta$. Then
 $\cos(A+B) = \cos(\theta+\theta) = \cos 2\theta \equiv \cos \theta \cos \theta - \sin \theta \sin \theta$
 $\equiv \cos^2 \theta - \sin^2 \theta$
 Therefore $\cos 2\theta \equiv \cos^2 \theta - \sin^2 \theta$

(ii) $\cos^2 \theta - \sin^2 \theta \equiv (1 - \sin^2 \theta) - \sin^2 \theta = 1 - 2\sin^2 \theta$
 So $\cos 2\theta \equiv 1 - 2\sin^2 \theta$

(iii) $\cos^2 \theta - \sin^2 \theta \equiv \cos^2 \theta - (1 - \cos^2 \theta) = 2\cos^2 \theta - 1$
 So $\cos 2\theta \equiv 2\cos^2 \theta - 1$

Q2

2

(i) Prove the identity

$$\sin 3\theta \equiv 3\sin \theta - 4\sin^3 \theta$$

(ii) Show by counter-example that

$$\cos 3\theta \equiv 3\cos \theta - 4\cos^3 \theta$$

$$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B \quad [\text{Compound angle formula}]$$

$$\sin 2A \equiv 2\sin A \cos A \quad [\text{Double angle formula}]$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A \quad [\text{Double angle formula}]$$

$$\sin^2 \theta + \cos^2 \theta \equiv 1 \Rightarrow \cos^2 \theta \equiv 1 - \sin^2 \theta$$

[5]

save my exams

(i) $\sin 3\theta = \sin(2\theta + \theta)$, so
 $\sin 3\theta \equiv \sin 2\theta \cos \theta + \cos 2\theta \sin \theta$
 $\equiv (2\sin \theta \cos \theta) \cos \theta + (1 - 2\sin^2 \theta) \sin \theta$
 $\equiv 2\sin \theta \cos^2 \theta + \sin \theta - 2\sin^3 \theta$
 $\equiv 2\sin \theta (1 - \sin^2 \theta) + \sin \theta - 2\sin^3 \theta$
 $\equiv 2\sin \theta - 2\sin^3 \theta + \sin \theta - 2\sin^3 \theta$
 $\equiv 3\sin \theta - 4\sin^3 \theta$
 Therefore $\sin 3\theta \equiv 3\sin \theta - 4\sin^3 \theta$

(ii) Let $\theta = 0$. Then
 $\cos 3\theta = \cos(0) = 1$
 $3\cos \theta - 4\cos^3 \theta = 3\cos(0) - 4\cos^3(0) = 3(1) - 4(1)^3 = -1$
 $1 \neq -1$
 Therefore $\cos 3\theta \not\equiv 3\cos \theta - 4\cos^3 \theta$

Q3

3

Show that

$$\cos 4\theta + \cos \frac{\pi}{3} \equiv 8 \sin^4 \theta - 8 \sin^2 \theta + \frac{3}{2}$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2 \cos^2 A - 1 \equiv 1 - 2 \sin^2 A \quad \text{[Double angle formula]}$$

$$\sin 2A \equiv 2 \sin A \cos A \quad \text{[Double angle formula]}$$

$$\sin^2 A + \cos^2 A \equiv 1 \Rightarrow \cos^2 A \equiv 1 - \sin^2 A$$

[5]

save my exams

$$\cos 4\theta = \cos(2(2\theta)), \text{ so}$$

$$\begin{aligned} \cos 4\theta &\equiv 1 - 2 \sin^2 2\theta \\ &\equiv 1 - 2(2 \sin \theta \cos \theta)^2 \\ &= 1 - 2(4 \sin^2 \theta \cos^2 \theta) \\ &= 1 - 8 \sin^2 \theta \cos^2 \theta \\ &\equiv 1 - 8 \sin^2 \theta (1 - \sin^2 \theta) \\ &= 1 - 8 \sin^2 \theta + 8 \sin^4 \theta \end{aligned}$$

Also $\cos \frac{\pi}{3} = \frac{1}{2}$, so

$$\begin{aligned} \cos 4\theta + \cos \frac{\pi}{3} &\equiv (1 - 8 \sin^2 \theta + 8 \sin^4 \theta) + \frac{1}{2} \\ &= 8 \sin^4 \theta - 8 \sin^2 \theta + \frac{3}{2} \end{aligned}$$

Therefore

$$\cos 4\theta + \cos \frac{\pi}{3} \equiv 8 \sin^4 \theta - 8 \sin^2 \theta + \frac{3}{2}$$

Q4

4

Prove that

$$\cot^2 \theta - \tan^2 \theta \equiv 4 \cot 2\theta \operatorname{cosec} 2\theta$$

$$\cot A \equiv \frac{\cos A}{\sin A} \quad \tan A \equiv \frac{\sin A}{\cos A}$$

$$\sin^2 A + \cos^2 A \equiv 1$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2 \cos^2 A - 1 \equiv 1 - 2 \sin^2 A \quad \text{[Double angle formula]}$$

$$\sin 2A \equiv 2 \sin A \cos A \quad \text{[Double angle formula]}$$

$$\cot A \equiv \frac{\cos A}{\sin A} \quad \operatorname{cosec} A \equiv \frac{1}{\sin A}$$

[5]

save my exams

$$\begin{aligned} \cot^2 \theta - \tan^2 \theta &\equiv \frac{\cos^2 \theta}{\sin^2 \theta} - \frac{\sin^2 \theta}{\cos^2 \theta} \\ &= \frac{\cos^4 \theta - \sin^4 \theta}{\sin^2 \theta \cos^2 \theta} \\ &= \frac{(\cos^2 \theta - \sin^2 \theta)(\cos^2 \theta + \sin^2 \theta)}{\sin^2 \theta \cos^2 \theta} \end{aligned}$$

$$\equiv \frac{\cos^2 \theta - \sin^2 \theta}{\sin^2 \theta \cos^2 \theta}$$

$$\equiv \frac{4(\cos^2 \theta - \sin^2 \theta)}{4 \sin^2 \theta \cos^2 \theta}$$

$$\equiv \frac{4(\cos^2 \theta - \sin^2 \theta)}{(2 \sin \theta \cos \theta)^2}$$

$$\equiv \frac{4 \cos 2\theta}{\sin^2 2\theta} = 4 \left(\frac{\cos 2\theta}{\sin 2\theta} \right) \left(\frac{1}{\sin 2\theta} \right)$$

$$\equiv 4 \cot 2\theta \operatorname{cosec} 2\theta$$

Therefore

$$\cot^2 \theta - \tan^2 \theta \equiv 4 \cot 2\theta \operatorname{cosec} 2\theta$$

Q5

5

Prove the identity

$$\frac{1 - \tan^2 x}{\cos 2x} \equiv \sec^2 x \quad x \neq \frac{2k+1}{4}\pi$$

$$\tan x \equiv \frac{\sin x}{\cos x}$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A \quad \text{[Double angle formula]}$$

$$\sec x \equiv \frac{1}{\cos x}$$

[5]

save my exams

$$\begin{aligned} \frac{1 - \tan^2 x}{\cos 2x} &\equiv \frac{1 - \frac{\sin^2 x}{\cos^2 x}}{\cos^2 x - \sin^2 x} \\ &\equiv \frac{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}}{\cos^2 x - \sin^2 x} \\ &\equiv \frac{\cancel{\cos^2 x} - \cancel{\sin^2 x} \cdot 1}{\cos^2 x (\cancel{\cos^2 x - \sin^2 x})} \\ &\equiv \frac{1}{\cos^2 x} \\ &\equiv \left(\frac{1}{\cos x}\right)^2 \\ &\equiv \sec^2 x \end{aligned}$$

Therefore

$$\frac{1 - \tan^2 x}{\cos 2x} \equiv \sec^2 x$$

Q6

6

Prove the identity

$$\operatorname{cosec} x \equiv \frac{\frac{1}{2} \sec^2 \frac{x}{2}}{\tan \frac{x}{2}}$$

$$\operatorname{cosec} \theta \equiv \frac{1}{\sin \theta}$$

$$\sin 2A \equiv 2 \sin A \cos A \quad \text{[Double angle formula]} \quad A = \frac{x}{2}$$

$$\sec \theta \equiv \frac{1}{\cos \theta} \quad \tan \theta \equiv \frac{\sin \theta}{\cos \theta}$$

[4]

save my exams

$$\begin{aligned} \operatorname{cosec} x &\equiv \frac{1}{\sin x} \\ &\equiv \frac{1}{2 \sin \frac{x}{2} \cos \frac{x}{2}} \\ &\equiv \frac{1/\cos^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2} / \cos^2 \frac{x}{2}} \quad \leftarrow \text{Divide Top and Bottom by } \cos^2 \frac{x}{2} \\ &\equiv \frac{\frac{1}{\cos^2 \frac{x}{2}}}{2 \left(\frac{\sin \frac{x}{2}}{\cos \frac{x}{2}}\right)} \\ &\equiv \frac{\sec^2 \frac{x}{2}}{2 \tan \frac{x}{2}} = \frac{\frac{1}{2} \sec^2 \frac{x}{2}}{\tan \frac{x}{2}} \end{aligned}$$

Therefore

$$\operatorname{cosec} x \equiv \frac{\frac{1}{2} \sec^2 \frac{x}{2}}{\tan \frac{x}{2}}$$

Q7

7

Show that

$$\tan \frac{x}{2} \equiv \frac{1}{\operatorname{cosec} x + \cot x} \quad x \neq 2k\pi$$

$$\operatorname{cosec} A \equiv \frac{1}{\sin A} \quad \cot A \equiv \frac{\cos A}{\sin A}$$

$$\sin 2A \equiv 2 \sin A \cos A \quad \text{[Double angle Formula]}$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2 \cos^2 A - 1 \equiv 1 - 2 \sin^2 A \quad \text{[Double angle Formula]}$$

$$\tan A \equiv \frac{\sin A}{\cos A}$$

[5]

save my exams

$A = \frac{x}{2}$

$$\begin{aligned} \frac{1}{\operatorname{cosec} x + \cot x} &\equiv \frac{1}{\frac{1}{\sin x} + \frac{\cos x}{\sin x}} \\ &= \frac{1}{\frac{1 + \cos x}{\sin x}} = \frac{\sin x}{1 + \cos x} \\ &\equiv \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{1 + (2 \cos^2 \frac{x}{2} - 1)} \\ &= \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \\ &= \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \\ &\equiv \tan \frac{x}{2} \end{aligned}$$

Therefore,

$$\tan \frac{x}{2} \equiv \frac{1}{\operatorname{cosec} x + \cot x}$$